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LETTER TO THE EDITOR

Exact multisoliton solution of the inhomogeneously broadened self-induced transparency equations

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Abstract. An exact solution of the inhomogeneously broadened self-induced transparency equations is given which describes the multiple collision of N solitons with different velocities. The solution is the same as the sharp-line solution except that the amplitude dependent velocity of each soliton has a different functional form. An exact N soliton solution of the form of the broadened Maxwell–Bloch equations valid at low densities is also reported.

Recently an N soliton solution of the self-induced transparency (SIT) equations has been discovered for the resonant sharp-line limit (Gibbon and Eilbeck 1972, Caudrey *et al* 1973a)†. In this letter we give N soliton solutions of the physically important inhomogeneously broadened equations; the solutions prove to be a natural extension of the sharp-line solutions. In dimensionless form the inhomogeneously broadened SIT equations are (McCall and Hahn 1969, Lamb 1971):

$$E_x(x,t)+E_t(x,t) = \alpha\langle P(\Delta\omega,x,t)\rangle \tag{1}$$

$$N_t(\Delta\omega,x,t) = -E(x,t)P(\Delta\omega,x,t) \tag{2a}$$

$$P_t(\Delta\omega,x,t) = E(x,t)N(\Delta\omega,x,t)+\Delta\omega Q(\Delta\omega,x,t) \tag{2b}$$

$$Q_t(\Delta\omega,x,t) = -\Delta\omega P(\Delta\omega,x,t), \tag{2c}$$

where for any $F(\Delta\omega)$

$$\langle F(\Delta\omega)\rangle = \int_{-\infty}^{+\infty} F(\Delta\omega')g(\Delta\omega')\mathrm{d}(\Delta\omega'). \tag{3}$$

The spectrum which characterizes the broadening, $g(\Delta\omega)$, is normalized such that

$$\int_{-\infty}^{+\infty} g(\Delta\omega')\mathrm{d}(\Delta\omega') = 1. \tag{4}$$

E is the envelope modulating a strictly *resonant* carrier wave, P and Q are the out-of-phase and in-phase components of the microscopic polarization, N is a measure of the atomic inversion, and α is a dimensionless constant proportional to the density of model two-level atoms. The boundary conditions for an attenuating medium are E, P, $Q \to 0$; $N \to -1$ as $x \to \pm\infty$, and the constant of integration, $N^2+P^2+Q^2$, is unity. In the derivation of the SIT equations (Lamb 1971) it is assumed that $g(\Delta\omega)$ and

† An equivalent solution of the closely related sine-Gordon equation has been given by Hirota (1972).

$Q(\Delta\omega, x, t)$ are even and odd functions respectively of $\Delta\omega$, so that $\langle Q \rangle = 0$. This choice makes the slowly varying phase a constant.

The main result of this letter is that the N soliton solution of the inhomogeneously broadened SIT equations (1), (2) is

$$E^2 = 4\frac{\partial^2}{\partial t^2} \ln f(x, t) \tag{5a}$$

$$f(x, t) = \det |M| \tag{5b}$$

where the $N \times N$ matrix M has the form

$$M_{ij}(x, t) = \frac{2(E_i E_j)}{E_i + E_j}\{\exp(\theta_i) + (-1)^{i+j} \exp(-\theta_j)\} \tag{6}$$

and

$$\theta_i = \omega_i t - \langle \kappa_i \rangle x + \delta_i \tag{7a}$$

$$\omega_i = \tfrac{1}{2} E_i \tag{7b}$$

$$\kappa_i = \omega_i[1 + 4\alpha\{E_i^2 + 4(\Delta\omega)^2\}^{-1}]. \tag{7c}$$

The E_i and δ_i are $2N$ arbitrary constants determining the amplitude and phase respectively of the ith soliton.

The proof that equations (5)–(7) are an exact solution of (1), (2) is as follows. First we consider the simpler set of equations obtained by replacing (1) by

$$E_x + E_t = \alpha P; \tag{8}$$

E is now dependent on $\Delta\omega$. Equation (8) can be derived *mathematically* from equation (1) by putting $g(\Delta\omega') = \delta(\Delta\omega - \Delta\omega')$ but equations (2) and (8) and their solutions cannot be interpreted *physically* as the sharp-line off-resonance case since $g(\Delta\omega')$ is no longer symmetric. The exact N soliton solution of these equations is the same as (5)–(7) (Caudrey *et al* 1973b) with equation (7*a*) becoming

$$\theta_i = \omega_i t - \kappa_i x + \delta_i. \tag{9}$$

With the solution to equations (2), (8) known, the solution to equations (1), (2) can be constructed as follows. The dependence of E in (8) as a function of $\Delta\omega$ is entirely contained in the formulae for the θ_i. Considering E as a function of the θ_i (and the parameters E_i) enables the equations (2), (8) to be written in the form

$$\sum_i (\omega_i - \kappa_i)\frac{\partial}{\partial\theta_i} E = \alpha P \tag{10a}$$

$$\sum_i \omega_i \frac{\partial}{\partial\theta_i} N = -EP \tag{10b}$$

$$\sum_i \omega_i \frac{\partial}{\partial\theta_i} P = EN + \Delta\omega Q \tag{10c}$$

$$\sum_i \omega_i \frac{\partial}{\partial\theta_i} Q = -\Delta\omega P. \tag{10d}$$

Mathematically we can consider $\Delta\omega$ and the θ_i as *independent* variables in equations (10). Since in equation (10*a*) the only dependence on $\Delta\omega$ is contained in κ_i and P, we can multiply by $g(\Delta\omega)$ and integrate over $\Delta\omega$ to get

$$\sum_i (\omega_i - \langle\kappa_i\rangle)\frac{\partial}{\partial\theta_i}E = \alpha\langle P\rangle. \tag{11}$$

It follows that the broadened SIT equations (1), (2) are satisfied by E as defined in (5)–(7). This completes the proof. The simplicity of the proof depends on the linearity of the Maxwell equation (Bullough and Ahmad 1971); the introduction of non-linearities in the Maxwell equation obviously causes difficulties with broadening (Matulic and Eberly 1972).

Once E is known N can be shown to be of the form

$$N = -1-2\alpha^{-1}\sum_{i,j}(\omega_i-\kappa_i)\omega_j\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln f \tag{12}$$

and P and Q can be calculated from (2*a*) and (2*b*). It can be shown that Q is an odd function of $\Delta\omega$ and hence satisfies our original condition.

The similarity of the solutions of the SIT equations with and without inhomogeneous broadening means that the properties of the sharp-line resonant solution ($g(\Delta\omega) = \delta(\Delta\omega)$) described in Gibbon and Eilbeck (1972) carry straight over to the broadened case, with the exception that the velocity of each soliton is now derived from $\langle\kappa_i\rangle$, instead of (7*c*) with $\Delta\omega = 0$. In particular the total phase shift from a multiple collision is the same linear sum of two soliton terms involving the E_i only. Equations (5)–(7) can be used to obtain broadened 0π and $2N\pi$ pulses in the same way as in the sharp-line limit.

Finally we note that our solution (5)–(7) is also an exact N soliton solution of the so called reduced Maxwell–Bloch (RMB) equations (Eilbeck *et al* 1973):

$$E_x + E_t = \alpha\langle r_t\rangle \tag{13a}$$

$$u_t = -Es \tag{13b}$$

$$s_t = \omega_s r + Eu \tag{13c}$$

$$r_t = -\omega_s s. \tag{13d}$$

Equation (13*a*) is an approximate form of the full Maxwell equation in which back-scattering is neglected. This approximation is valid at sufficiently low densities (Eilbeck *et al* 1973, Eilbeck 1972). Equations (13*b*)–(13*d*) are the normal Bloch-type equations for a two-level atom system with resonant frequency ω_s. Although equations (2), (8) and equations (13) are mathematically equivalent, an important physical difference is that (2), (8) describe the evolution of an *envelope* modulating a carrier wave whereas (13) describes the *field*. The RMB equations are inhomogeneously broadened in the same way as the SIT equations but with a distribution $g(\omega_s)$ in the resonance frequency instead of in the off-resonance parameter $\Delta\omega$.

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